# AUDIO NONLINEAR MODELING THROUGH HYPERBOLIC TANGENT FUNCTIONALS

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## ABSTRACT

In the present contribution we present the preliminary results of a black box nonlinear system (NLS) modeling. The NLS is composed by a nonlinear sigmoid-type input-output relationship (NLTF) followed by a linear system (LTI), as in a Hammerstein nonlinear system. Here, the used NLTF is derived from a deformation of the Hyperbolic Tangent power expansion. The advantage of using the hyperbolic tangent function is that nonlinearity depends on the linear and cubic terms that measure curvature (and thus nonlinearity) of the transfer function. The hyperbolic tangent model is extended to other types of nonlinear systems by expanding the nonlinear system in linear and increasingly nonlinear contributions, where the expansion parameters are deformed to enhance or suppress specific nonlinear modes of the expansion. Simulations were performed using Matlab 2012a. The preliminary results show fairly good agreement between the system obtained by parametric inference and a reference system, with mean square error (MSE)=0.035.

## 1. INTRODUCTION

Linear Time Invariant Systems (LTI) have been extensively studied for decades [1], [2]. However, the nonlinearities in audio musical systems that are responsible for specific tone characteristics desired by many musicians [3], [4].

The nonlinearities of the NLS can be weak (i.e the NLS can be represented by a power series expansion of only a few terms) or strong (higher order terms of the power series expansion must be calculated for the modelling system) [5]. If the NLS is very weak, a linear approximation can be used. On the other hand, very strong nonlinear systems as samplers, switches and other systems with discontinuities in the system representation must use the entire terms of its power series representation, which is rather inconvenient for modeling purposes [6]. Then, in accordance to the "strength" of the nonlinearities an appropriate and efficient technique to be used is in order. Guitar and bass vacuum-tube amplifiers can be considered weakly NLS and most of NLS identification techniques make this assumption.

By its own nature, when a weak nonlinear system related with saturation (NLS) has as input a pure sine wave $x(t) = A_1 \cos(2\pi f_1 t + \phi_1)$, higher harmonics related to the input frequency will appear in the output, according with the relation $y(t) = \sum_n B_n \cos(2\pi n f_1 t + \phi_n)$. Accordingly, LTI identification techniques as impulse response, convolution and Laplace or Fourier analysis cannot be used with NLS. Hence, many attempts have been suggested to model such NLS [4].

The main techniques for NLS identification/modeling can be classified in **black box approaches**, **white box approaches** or **analytic modeling techniques** and **grey box approaches**. Black box approaches involve no *a priory* knowledge on the NLS to be modelled/identified. The NLS is excited with a set of test signals and using obtained outputs, the coefficients of a polynomial or power series relation are estimated in such a way as to minimize the error between the true NLS system and the model. In this category one finds the Volterra, Volterra-Wiener and Hammerstein model techniques, for instance, [5], [7], [8], [9], [10], [11], [3], [12], [13]. On the other hand, white box approaches involve total knowledge of the system to be modelled. To achieve this, one needs to know the circuit theory and the schematics of the devices to be modelled. In possession of this information, the nonlinear differential equation set of the circuits involved can be obtained and solved. In this category the models for SPICE Simulation, transient modified nodal analysis, state-space representation and numerical methods for solving nonlinear circuits techniques can be found as for instance in refs. [14], [15], [16], [17], [18], [19], [20]. Grey box approaches use polynomial models as in black box techniques, but incorporating some knowledge about the nonlinear circuits used. Good reviews of NLS modeling techniques can be found in [4], [21].

In this work we propose a black box method where the NLS is composed by a nonlinear sigmoid-type input-output relationship (nonlinear transfer function, NLTF) followed by a LTI, as in Hammerstein nonlinear systems, [12]. This is shown in Fig. 1. The first section shows the calculation of the output from the entire system given a generic input. Then the developments based on the Hyperbolic Tangent series and its coefficients estimation are shown, followed by a practical example.

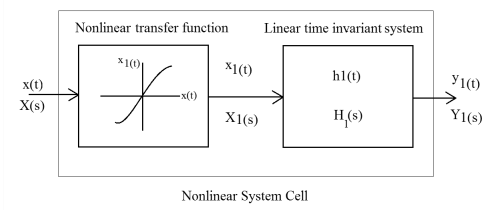


Figure 1: *The entire NLS proposed.*

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## 2. A MODEL PROPOSAL FOR THE MODELING SYSTEM

The NLS model (or cell) proposed here (Fig. 1) is half the system proposed in [3] and is appropriate to model weakly NLS as a vacuum tube amplifiers pre-amplifier or power stages and distortion and overdrive devices. It is composed by an NLTF followed by an LTI system as in a Hammerstein model [12]. The modeling procedure consists in applying appropriate test signals in order to estimate both the NLTF and the LTI system that minimize the error between the output of the NLS to be modeled and the NLS model, by some minimization criterium of the error between the true output signal against the NLS model output. Here we used for simplicity the standard minimum square method as specified further down in eq. (10). The modeling procedure is shown in Fig. 2. One of the differences of this work is that the NLTF used

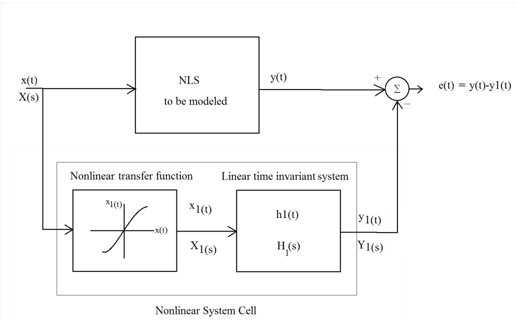


Figure 2: *The modeling system and its variables.*

is derived from a deformation of the Hyperbolic Tangent power expansion. The development and identification of the coefficients of this deformed expansion will be explained further below. As a matter of fact, the harmonics produced for the NLTF are filtered by the LTI system. This circuit section usually models the frequency response of the amplifier stage to be modeled, including the parasitic capacitances, tone controls or a simplified output transformer and speaker model.

Generally speaking, many LTI have the transfer function in Laplace domain described as a quotient of polynomials by ([2], [22]) :

$$H_1(s) = \frac{Y_1(s)}{X_1(s)} = G\frac{\sum_{k=0}^{M} B_k s^{M-k}}{\sum_{k=0}^{N} A_k s^{N-k}}\,, \tag{1}$$

where $G$ is the Global Gain of the LTI and $B_k$ and $A_k$ are the coefficients of the differential equation that rules the LTI.

It is straight forward to show [2] that the differential equation which relates the output $y_1(t)$ with the input $x_1(t)$ is

$$\sum_{k=0}^{N} A_k \frac{\mathrm{d}^{N-k} x_1(t)}{\mathrm{d}t^{N-k}} = G\sum_{k=0}^{M} B_k \frac{\mathrm{d}^{M-k} y_1(t)}{\mathrm{d}t^{M-k}}\,, \tag{2}$$

recalling that $x_1(t)$ is the response of the NLS to an input $x(t)$.

The impulse response of eq. (1) can be analytically evaluated by finding the Poles (roots of the denominator polynomial) of the LTI, expanding eq. (1) in Partial Fractions and finding the inverse Laplace transform to the expansion [2],[22]. Otherwise, a numerical algorithm can be applied to eq. (2) to find a solution for that differential equation.

The output $y_1(t)$ can be represented by the convolution

$$y_1(t) = \int_{-\infty}^{\infty} h_1(\tau) x_1(t-\tau)\,\mathrm{d}\tau\,, \tag{3}$$

where $x_1(t)$ is the output of the NLTF for an input $x(t)$, or formally $x_1(t) = NLTF|_{x(t)}(t)$, so that eq. (3) is

$$y_1(t) = \int_{-\infty}^{\infty} h_1(\tau) NLTF|_{x(t)}(t-\tau)\,\mathrm{d}\tau\,. \tag{4}$$

For instance, if one estimate the NLTF as an arc hyperbolic sine function, like the one that mimics diode distortion pedals and a JCM900 preamp output voltage (ref. [15]), then eq. (4) can be written as

$$y_1(t) = \int_{-\infty}^{\infty} h_1(\tau) \mathrm{arcsinh}|_{x(t)}(t-\tau)\,\mathrm{d}\tau\,. \tag{5}$$

To estimate this sub-system one has to apply an input signal $x(t)$ small enough in order that the NLS to be modeled may be considered a linear system. This is fairly true for many amplifiers and distortion devices, and all techniques already developed for LTI can be used, in time or in frequency domain. In the present case the identification/estimation is performed by $\widetilde{H_1}(\omega) = Y(\omega)/X(\omega)$.

## 3. THE NONLINEAR SUB-SYSTEM MODEL

As a starting point for modeling the weakly nonlinear properties of an audio-system, we start from a mathematical function, i.e. the hyperbolic tangent

$$\begin{aligned} x_1 &= \tanh\alpha \\ &= \frac{e^{\alpha} - e^{-\alpha}}{e^{\alpha} + e^{-\alpha}}\,, \end{aligned} \tag{6}$$

that in certain limits of $\alpha$ (the input signal and amplification) exhibits predominantly linear properties and beyond these limits, then the full nonlinear characteristics as can be seen in Fig.3. The

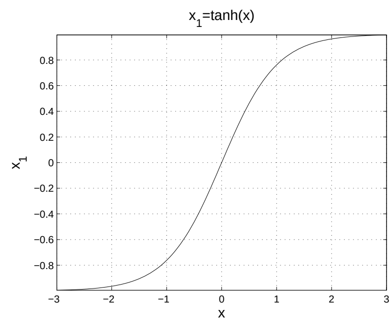


Figure 3: $x_1 = \tanh x$.

choice for this specific function resides in the fact that a representation by exponential functions may be easily implemented computationally. Then, the normalized output signal $y_1(t)$ of the nonlinear sub-system may be described by the input signal $x(t)$ and the response of the nonlinear sub-system

$$x_1(t) = \tanh\left(\underbrace{\kappa(x(t) + x_0)}_{\alpha}\right) - \tanh(\kappa x_0)\ , \tag{7}$$

where $\kappa$ plays the role of an amplification factor and for a symmetric NLTF $x_0 \equiv 0$, whereas for asymmetric NLTF $x_0 \neq 0$. Note that besides the translation of the argument of the hyperbolic function there is also necessarily a shift in the function such as to match a zero input signal with a zero output signal. In this kind of approach the nonlinearity is a unique one which may be seen from the constituent differential equation that has the hyperbolic tangent as solution.

$$\frac{\partial^2 \xi(\alpha)}{\partial \alpha^2} = 2\left(\xi^3(\alpha) - \xi(\alpha)\right) \tag{8}$$

The "beauty" of the hyperbolic tangent function is that the second derivative and its dependence on the linear and cubic signal is a direct measure for the curvature of the transfer function and thus for the nonlinear behavior. In order to extend this nonlinear model to other types of nonlinear systems we proceed in two steps, first expand the nonlinear system in linear and increasing nonlinear contributions, and second, extend the expansion by introducing new parameters that allow to tune the balance of the linear and nonlinear character, in other words, allow to enhance or suppress specific nonlinear modes of the expansion. For signals $x(t)$ sufficiently small most of the audio systems show a linear behavior, which corresponds to the first term of a small signal expansion, where including nonlinear terms nevertheless restricting generality we truncate at the fifth term that represents a nonlinear character. If $x_1 = \tanh(\kappa x)$ with $\kappa x = \alpha$, then

$$\begin{aligned}
\xi(\alpha) &= \left( \sum_{n=0}^{\infty} \underbrace{\frac{1}{n!} \frac{\partial^n \tanh(\alpha)}{\partial \alpha^n}\Big|_{\alpha=\kappa x_0}}_{a_n} \alpha^n \right) - \tanh(\kappa x_0) \\
&= \left(1 - \tanh^2(\alpha)\right)\Big|_{\alpha=\kappa x_0} \alpha \\
&\quad - \tanh(\alpha)\left(1 - \tanh^2(\alpha)\right)\Big|_{\alpha=\kappa x_0} \alpha^2 \\
&\quad - \left(\frac{1}{3} - \tanh^2(\alpha)\right)\left(1 - \tanh^2(\alpha)\right)\Big|_{\alpha=\kappa x_0} \alpha^3 \\
&\quad + \left(\frac{2}{3} - \frac{1}{2}\tanh^2(\alpha)\right)\tanh(\alpha) \times \\
&\qquad\qquad \times \left(1 - \tanh^2(\alpha)\right)\Big|_{\alpha=\kappa x_0} \alpha^4 \\
&\quad + \left(\frac{2}{15} - \frac{21}{30}\tanh^2(\alpha) + \frac{1}{2}\tanh^4(\alpha)\right) \times \\
&\qquad\qquad \times \left(1 - \tanh^2(\alpha)\right)\Big|_{\alpha=\kappa x_0} \alpha^5 \\
&\quad + \mathcal{O}(\alpha^6)
\end{aligned} \tag{9}$$

It is noteworthy that all terms of the expansion are linearly independent so that one may modify the original factors $a_n \to a_n + \delta a_n$ by an increment or decrement $\delta a_n$ and thus adjust the linear to the nonlinear proportions beyond that given by the hyperbolic tangent function and additionally shape the nonlinearity since $[2\left(y^3(\alpha) - y(\alpha)\right)]$ is no longer the original curvature.

Recalling that we are considering a black box modeling, nevertheless focus on specific audio system characteristics the connection between a system with measured total response function (microphone, amplifier and speaker characteristics) may be determined using parametric inference techniques as laid out for instance in [23]. To this end the input signal $x(t)$ as well as the desired output signal $y(t)$ is discretized in $T$ times $t_i$ ($i \in \{0, \dots, T\}$) and the factors $\delta a_n$ are adjusted such as to minimize the difference of the model and the experimental data.

$$\min_{\{x_0, \{\delta a_n\}_{n=1}^N\}} \left( \sum_{i=0}^{T} \left\| y(t_i) - \int_0^{t_i} h(t_i - \tau)\xi(\tau)\, d\tau \right\| \right) \tag{10}$$

Here $||\cdot||$ denotes any convenient norm or semi-norm. In this work we use the Euclidian norm.

A comment is in order here, other approaches make use of orthogonal functional spaces which they use to model nonlinear responses [5]. With respect to this approach ours is not that much different, however, the advantage of the present approach is justified by the fact that all derivatives of the hyperbolic tangent function may again be represented by hyperbolic tangent functions and constants, in other words only one function (represented by exponential functions) is needed to generate the whole expansion, where in the present discussion we showed the linear and five nonlinear terms that appear in the expansion. In the symmetric case the parity even terms in $\alpha$ disappear identically. Moreover, in principle the tuning needs only to be stored in a vector that indicates directly the linear and nonlinear characteristics to be modeled. For example, if the NLTF would be a simple hyperbolic tangent, such as $y_1(t) = \tanh x(t)$, the hyperbolic tangent can be expressed as a power series by [24]

$$\tanh x = x - \frac{x^3}{3} + 2\frac{x^5}{15} - 17\frac{x^7}{315} + \dots, \tag{11}$$

and then the coefficients $a_n$ for n even would be 0 and $a_1 = 1$, $a_3 = -\frac{1}{3}$, $a_5 = \frac{2}{15}$, $a_7 = -\frac{7}{315}$ and so on. For a case of arc hyperbolic sine as the NLTF, expanding the arc hyperbolic sine gives [24]

$$\text{arcsinh} = x - \frac{x^3}{6} + 3\frac{x^5}{40} + \dots, \tag{12}$$

and in this case $a_n$ for n even would be 0 and $a_1 = 1$, $a_3 = -\frac{1}{6}$, $a_5 = \frac{3}{40}$ and so on, which is $\delta a_3 = -\frac{1}{2}a_3$ and $\delta a_5 = -\frac{7}{120}$ with respect to the hyperbolic tangent case.

## 4. METHODS

In order to simulate a system to be identified, all the signals and systems were performed using Matlab 2012a version. The sampling frequency used is $f_s = 120000$ samples per second in order to accommodate the harmonics of the output signal up to the $5^{th}$ component without aliasing, simulating an A/D system with $f_s = 20000$ samples per second with an anti-alias analogue $6^{th}$ order low-pass filter with cut-off frequency of 10kHz before the sampling process and after the discretization, the signal is upsampled 6 times. But all the signals involved to be exhibited are downsampled to $f_{s1} = 20000$ samples per second.

For the NLTF to be modeled, we chose the distortion simulation presented in [21], given by

$$x_1 = \text{sign}(x)(1 - e^{-|x|}). \tag{13}$$

The graphic of this NTFS is shown in Fig.4. The output of this block feeds a discrete linear system which is a $2_{nd}$ order digital Butterworth bandpass filter with cut-off frequencies of 100 Hz and 8000 Hz, normalized to radians. This simulates approximately the frequency response of a $12''$ guitar speaker. The frequency response of this system is shown in Fig.5: The signal $x[n]$ chosen for the LTI identification is a linear chirp from 0Hz at $n = 0$ to 10kHz at $n = 65535$ (or $t = 0.96$ s). In order to keep the nonlinear effects of the NLTF negligible, the signal must be so small that its amplitude does not surpass the linear part of the NLFT. Because of that, the maximal amplitude of the signal was chosen equal 0.01. The periodogram and the spectrogram of this signal is shown in Fig. 6.

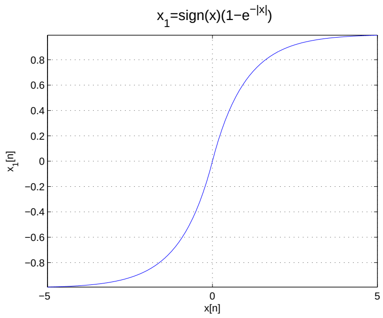


Figure 4: *Graphic of the NLTF* $x_1 = \text{sign}(1 - e^{-|x|})$

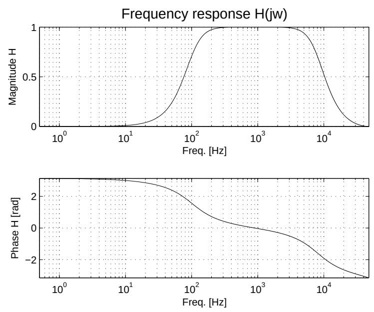


Figure 5: *Frequency response of the LTI system to be identified.*

Injecting the signal $x[n]$ in the system to modeled, the signal $y[n]$ is obtained and assuming the NLTF nonlinear effect is negligible, then $H_1(w)$ can be estimated by

$$\widetilde{H_1}[k] = \frac{Y[k]}{X[k]}, \tag{14}$$

where $Y[k]$ and $X[k]$ are the FFTs of the signals $x[n]$ and $y[n]$, respectively. In Fig. 7, can be seen the periodogram and the spectrogram of the signal y[n]. Note that using these signal amplitude, aliasing practically did not occur and the nonlinear effect appears as a very faint $3_{rd}$ harmonic line in its spectrogram. Just for the sake of exemplification, if the chirp signal used here would have a large amplitude $A$ (eg. $A = 5$), the output of the NLS to be modeled would had the periodogram and spectrogram shown in Fig.8 The frequency response of $\widetilde{H_1}[k]$ obtained is shown in Fig.9: The impulse response estimation of $\tilde{h_1}[n]$ is obtained performing the IFFT over $\widetilde{H_1}[k]$. It's worth noting that this form of estimation will provide a FIR system model, even if the LTI system to be estimated be an IIR system (as our case). But if one choose the appropriate number of coefficients for the FIR system, the error can be made negligible and the computational effort minimized. Using $N = 65536$, then the estimation of the LTI system $\widetilde{H_1}[k]$ impulse response $\tilde{h_1}[n]$ is obtained. The next step is to estimate the NLTF of the system. To do that, a second signal $x[n]$ is injected in the system, now a sine signal with fixed frequency inside the band pass of the system (already identified) but with amplitude large enough so the output signal $y[n]$ be distorted. Here the signal was a sine signal with frequency of 500Hz (around the center frequency of the Bandpass) and amplitude $A$ of 5. The input and output signals can be seen in Fig.12 To remove the phase influence of the LTI sub-system in the output signal, an inverse filter (aka equalizer filter) $g[n]$ obtained by turning the magnitude of

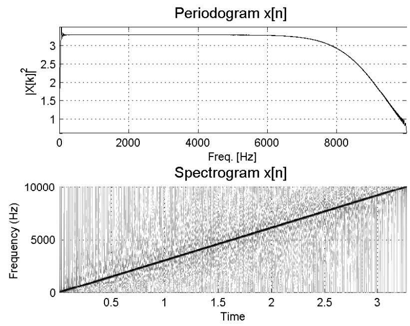


Figure 6: *The Periodogram and the Spectrogram of the signal x[n] for LTI estimation.*

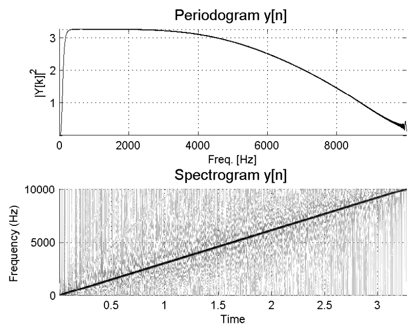


Figure 7: *The Periodogram and the Spectrogram of the signal y[n].*

$\widetilde{H_1}[k] = 1.0$ and then $g[n] = IFFT[\frac{1}{\widetilde{H_1}[k]}]$ as shown in Fig. 10. This will preserve the amplitude of the signal $y[n]$ but will correct the phase only. is applied to the output signal $y[n]$, and giving an equalized signal $y_{eq}[n]$. The scatter plot between $x[n]$ and $y_{eq}[n]$ is shown in Fig.11: As can be seen by the Lissajous curves obtained, the phase wasn't completely corrected but the phase error of $y_{eq}[n]$ is small enough for the next step. Finally, with both signal $x[n]$ and $y_{eq}[n]$, the new coefficients $a_k + \delta a_k$ of eq. (9) can be estimated. Here, we used minimization with Euclidian norm and consequently, least squares fitting techniques, evaluated up to the $10^{th}$ order. The coefficients obtained for our example are

$$\begin{aligned}
a_0 + \delta a_0 &= 0.000412543580279774\ , \\
a_1 + \delta a_1 &= 0.695903675375691\ , \\
a_2 + \delta a_2 &= 8,42100337873921e-06\ , \\
a_3 + \delta a_3 &= -0.0873565648035508\ , \\
a_4 + \delta a_4 &= -1.51387407429003e-05\ , \\
a_5 + \delta a_5 &= 0.00715137611726394\ , \\
a_6 + \delta a_6 &= 2.73945820531464e-06\ , \\
a_7 + \delta a_7 &= -0.000280736782949899\ , \\
a_8 + \delta a_8 &= -1.67046480776686e-07\ , \\
a_9 + \delta a_9 &= 4.11012487805344e-06\ , \\
a_{10} + \delta a_{10} &= 3.17102262802150e-09\ , \\
\cdots &.
\end{aligned}$$

The comparison between the outputs of the actual system and the NLS estimated is shown in Fig.12.

Finally, it is shown on Fig.13 the output of the system to be modeled and the model for all the parameters above, having as input a sine signal of $1kHz$.

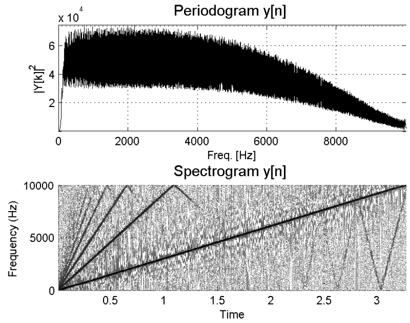


Figure 8: *The Periodogram and the Spectrogram of the signal y[n] for a chirp with* $A = 5$.

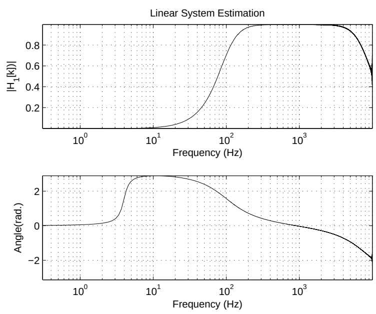


Figure 9: *Frequency response of the LTI system estimated.*

The Mean Square Error (MSE) between $y[n]$ and $y_1[n]$, estimated by

$$MSE = \frac{1}{N}\sum_{n=0}^{N-1}(y[n] - y_1[]n)^2 \tag{15}$$

was around $MSE = 0.035$.

## 5. CONCLUSIONS

In the present work we discussed a black box approach for a NLS that shall simulate the response of an audio system such as a tube amplifier for musical instruments with its characteristic frequency response of the amplifier, including the parasitic capacitances, tone controls, output transformer and speaker. To this end we used a chirp signal to excite the NLS followed by a LTI and adjusted the coefficients of a deformed hyperbolic tangent power expansion in order to reproduce a desired output by the use of parametric inference technique. The choice for the deformed hyperbolic tangent function resides in the fact that a representation by exponential functions may be easily implemented digitally and allows to describe symmetric as well as asymmetric amplification using translations of the argument and amplitude respectively. Moreover the hyperbolic tangent function has a simple relation to its nonlinearity since the second derivative depends on the linear and cubic function. As a specific example we used an exponential distortion $x_1 = \text{sign}(1 - e^{-|x|})$ followed by a Butterworth $2_{nd}$ order bandpass filter with cut-off frequencies related to 100Hz and 8kHz to simulate the frequency response of an 12" speaker. For the example used, the Mean Square Error (MSE) between $y[n]$ and $y_1[n]$ was around $MSE = 0.035$.

The authors of the present work are aware of the fact that there are other approaches similar to the present one, however, they make use of orthogonal functional spaces which they use to model nonlinear responses. We believe, that the advantage of the present approach is justified due to the fact that all derivatives of the hyperbolic tangent function may again be represented by hyperbolic tangent functions and constants. The deformation parameters that were determined to reproduce the desired input - output signal relation showed that the desired system can be reproduced with fairly good fidelity. In future work we intend to apply the proposed procedure to a selection of other amplifiers obtained by measurements and compare quality as well as computational efficiency for simulation applications.

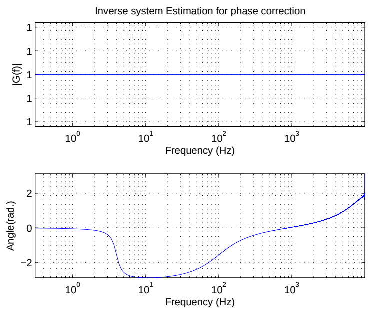


Figure 10: *Frequency response of the Inverse filter estimated.*

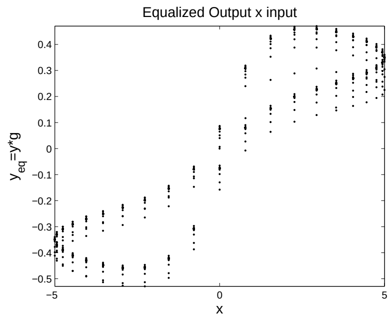


Figure 11: *Scatter plot between* $x[n]$ *and* $y_{eq}[n]$.

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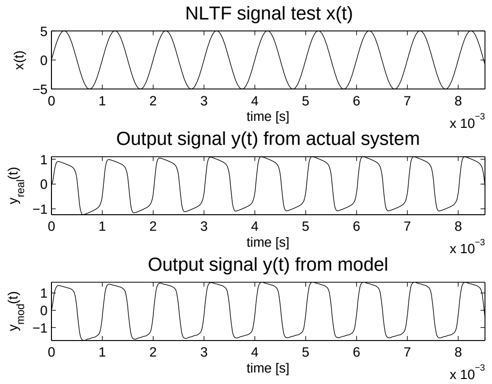


Figure 12: *Comparison between the outputs from the actual system and from the NLS estimated.*

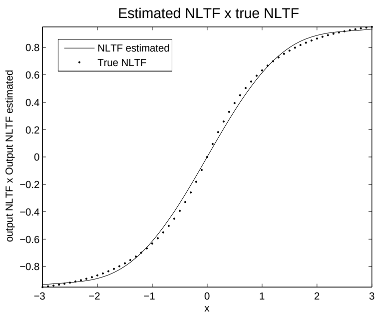


Figure 13: *Estimated NLTF vs. Actual NLTF.*

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